A NOTE ON CALCULATING AUTOCOVARIANCES OF PERIODIC *ARMA* MODELS

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ABSTRACT

An analytically simple and tractable formula for the start-up autocovariances of periodic $ARMA$ ($PARMA$) models is provided.

Keywords: $PARMA$ models, autocovariance functions, periodic Yule-Walker equations.

2000 Mathematics Subject Classification. Primary: 62M10; Secondary: 62F15.

1. INTRODUCTION

Autocovariance calculation procedures for $PARMA$ models are generally carried out using the periodic Yule Walker equations (e.g. Bentarzi and Aknouche, 2005). This approach has been considered earlier by Li and Hui (1988) for calculating $PARMA$ autocovariances, where the $(p+1)$-start-up autocovariances, for all seasons, were given through a matrix equation $A\gamma = y$, which is solved for γ (γ being the $(p+1)S\times 1$-vector of the start-up autocovariances). The latter equation is however analytically and computationally intractable since the matrix A is not given explicitly but formed through an appropriate algorithm. Adopting the same approach, Shao and Lund (2004) showed that the r-start-up ($r = \max(p,q)+1$) autocovariances may be obtained by solving a linear system $\Gamma U\gamma = \kappa$ for γ, where Γ and U are matrices of dimensions $rS \times (p+1)rS$ and $(p+1)rS \times rS$, respectively. While these matrices are given explicitly, the method remains relatively cumbersome since it requires an increasing bookkeeping due to the matrix product. This note proposes an improved computation procedure for calculating the $PARMA$ autocovariances based on the latter approach.

The proposed method computes the $(p+1)$-start-up autocovariances based on a linear system with a corresponding matrix given explicitly, whose analytical form exhibits a circular structure, naturally assorted with the model periodicity.

2. THE METHOD

Consider a causal $PARMA$ model of orders (p,q) and period S

$$\sum_{j=0}^{p}\phi_j^{(v)}y_{v-j+nS} = \sum_{j=0}^{q}\theta_j^{(v)}\varepsilon_{v-j+nS},\ 1\leq v\leq S,\ n\in\mathbb{Z}, \tag{1}$$

where $\phi_0^{(v)}=\theta_0^{(v)}=-1$ and $\{\varepsilon_t, t\in Z\}$ is a periodic white noise process, i.e., a sequence of uncorrelated random variables with mean zero and variance $E(\varepsilon_{v+nS}^2)=\sigma_v^2$, for $1\leq v\leq S$ and $n\in\mathbb{Z}$.

Let $\gamma_h^{(v)}=E\left(y_{v+nS}y_{v+nS-h}\right)$ be the autocovariance function at season v and lag $h\in\mathbb{Z}$. Then, it is well known (Li and Hui, 1988; Shao and Lund, 2004) that multiplying (1) by y_{v+nS-h} and tacking expectation, the $\left(\gamma_h^{(v)}\right)$ are completely identified from the difference equation

$$\gamma_h^{(v)}-\sum_{j=1}^{p}\phi_j^{(v)}\gamma_{h-j}^{(v-j)} = -\sum_{j=h}^{q}\theta_j^{(v)}\psi_{j-h}^{(v-h)}\sigma_{v-j}^2\mathbf{1}_{[h\leq q]},\ h\geq 0, \tag{2}$$

where the normalized cross-autocovariances $(\psi_k^{(v)})$, coefficients of the unique causal representation of the $PARMA$ process $\{y_t, t\in Z\}$, are given by (see e.g. Lund and Basawa, 2000; Shao and Lund, 2004)

$$\psi_k^{(v)} = -\theta_k^{(v)}\mathbf{1}_{[k\leq q]}+\sum_{j=1}^{\min(k,p)}\phi_j^{(v)}\psi_{k-j}^{(v-j)},\qquad k\geq 1,\ v=1,...,S, \tag{3}$$

with $\psi_0^{(v)}=1$ ($\mathbf{1}_{[.]}$ stands for the indicator function).

Equation (2) needs to be started from the knowledge of $\gamma_h^{(v)}$, $0\leq h\leq p$ and $1\leq v\leq S$. Once these start-up values are given, the $\gamma_h^{(v)}$ for $h>p$ may be obtained recursively from (2) while invoking (3). For $\gamma_h^{(v)}$ with negative lags, we may use the well-known relation $\gamma_{-h}^{(v)}=\gamma_h^{(v+h)}$. The main result of this note is to formulate a linear system for computing the $p+1$ necessary starting autocovariances. Let $\gamma=(\gamma_0^{(1)},...,\gamma_p^{(1)},\gamma_0^{(2)},...,\gamma_p^{(2)},...,\gamma_{0,}^{(S)},...,\gamma_p^{(S)})'$

be the $S(p+1)$-vector of such values and ζ be the $S(p+1)$-vector whose entries $\zeta_{hS+v} = \sum_{j=h}^{q} \theta_j^{(v)} \psi_{j-h}^{(v-h)} \sigma_{v-j}^2$, for $1 \le v \le S$ and $0 \le h \le p$, are the right-hand sides of (2).

Define the $(p+1)$-square matrices $\varphi_h^{(v)}$ $(h \ge 0,\ 1 \le v \le S)$ and $\mathbf{\Phi}_k^{(v)}$ $(k,\ v \in \{1, ..., S\})$ as follows

$$\varphi_h^{(v)} = \begin{cases} \left(\begin{array}{ccccc|c} \multicolumn{5}{l|}{\mathbf{0}_{h\times(p+1-h)}} & \mathbf{0}_{h\times h} \\ \hline -\phi_h^{(v)} & -\phi_{h+1}^{(v)} & \cdots & & -\phi_p^{(v)} & \\ 0 & -\phi_h^{(v)} & 0 & \cdots & 0 & \\ & \ddots & \ddots & \ddots & \vdots & \mathbf{0}_{(p+1-h)\times h} \\ \vdots & & & \ddots & 0 & \\ 0 & \cdots & & 0 & -\phi_h^{(v)} & \end{array}\right), & \text{for } h = 0, ..., p \\ \mathbf{0}_{(p+1)\times(p+1)} & \text{for } h \ge p+1, \end{cases} \tag{4}$$

$$\mathbf{\Phi}_k^{(v)} = \sum_{n \ge 0} \varphi_{nS+k}^{(v)},\ \ v, k \in \{1, ..., S\}, \tag{5}$$

and the $S(p+1)$-square matrix

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Phi}_0^{(1)} & \mathbf{\Phi}_{S-1}^{(1)} & \cdots & \mathbf{\Phi}_2^{(1)} & \mathbf{\Phi}_1^{(1)} \\ \mathbf{\Phi}_1^{(2)} & \mathbf{\Phi}_0^{(2)} & \cdots & \mathbf{\Phi}_3^{(2)} & \mathbf{\Phi}_2^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{\Phi}_{S-2}^{(S-1)} & \mathbf{\Phi}_{S-3}^{(S-1)} & \cdots & \mathbf{\Phi}_0^{(S-1)} & \mathbf{\Phi}_{S-1}^{(S-1)} \\ \mathbf{\Phi}_{S-1}^{(S)} & \mathbf{\Phi}_{S-2}^{(S)} & \cdots & \mathbf{\Phi}_1^{(S)} & \mathbf{\Phi}_0^{(S)} \end{pmatrix}.$$

where $\mathbf{0}_{m\times n}$ denotes the null matrix of dimension $m \times n$. Then, the starting autocovariance vector γ is the unique solution of the following linear system

$$\mathbf{\Phi}\gamma = \zeta, \tag{6}$$

whenever model (1) is causal. Note that, in view of (4), the infinite sum in (5) contains only p non zero terms. It may be possible to reduce the complexity of forming $\mathbf{\Phi}$ using its circular property. Indeed, equation (5) may be used to only evaluate the first bloc $\mathbf{\Phi}_k^{(1)}$, $k = 1, ..., S$. The blocs $\mathbf{\Phi}_k^{(v)}$ $(v = 2, ..., S)$ would be deduced from $\mathbf{\Phi}_k^{(1)}$ by substituting the corresponding parameters $\phi_j^{(1)}$ by $\phi_j^{(v)}$ for $j = 1, ..., p$.

3. CONCLUDING REMARKS

Despite the simplicity of the proposed method, it has the drawbacks that the starting autocovariances for all seasons are computed in the same bloc, thereby requiring $O\left((S(p+1))^3\right)$ operations, which might be very costly for models with a large period. This is the main limitation of the periodic Yule Walker approach compared to which the methods that compute autocovariances for distinct seasons separately (e.g. Aknouche, 2007) are more suitable.

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